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SHOCK WAVES IN ARBITRARY FLUIDS

(by Hermann Weyl)

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A Note Submitted
by the
Applied Mathematics Group, New York University
to the
Applied Mathematics Panel
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This note is of a theoretical character. Its results are not meant to lead to immediate applications; however, the subject is closely connected with topics that have arisen in connection with AMP Study 38 on shock waves.

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TABLE OF CONTENTS

Page

Summary ---------- II

Figs. 0, 1, 2 ---------- IV

Figs. 3, 4, 5 ---------- V

Figs. 6, 7 ---------- VI

Part I. Thermodynamics and the Shock Phenomenon ---------- 1-19

1. Shocks ---------- 1

2. Thermodynamical assumptions ---------- 4

3. The fundamental inequality for the direction $\overrightarrow{Z_0Z_1}$ - 9

4. Entropy and parametrization of the Hugoniot Line - 12

5. The limit of pressure along the Hugoniot Line ---- 15

6. The example of the ideal gas ---------- 16

Part II. Reflection of a Shock Wave ----------19-29

7. The algebra of reflection ---------- 19

8. Why glancing incidence is impossible ---------- 24

Part III. The Problem of the Shock Layer ----------30-41

9. Formulation of the problem ---------- 30

10. Character of the two singular points Z_0, Z_1 : Node and saddle ---------- 34

SUMMARY

The following investigation tries to clear up the general hydrodynamical and thermodynamical foundations of the shock phenomenon. The main results are contained in Part I, §§1-4, and Part III, §§9-10. The first part answers the question: What are the conditions for the equation of state of a fluid under which shocks with their distinctive qualitative features may be produced. These conditions, enumerated in §2, are partly of differential, partly of global nature. Part III investigates the physical structure of the shock layer whose "infinitesimal" width is of the order of magnitude ϵ provided heat conductivity and viscosity are small of the same order. Initial state and final state are singular points for the differential equations of the shock layer, and it is shown that they are of such a nature as to make one expect the problem to have a unique solution.

§§5-8 are only loosely connected with the main issue. The author's previous note, "A scheme for the computation of shock waves in gases and fluids", dealt with ideal gases for which the specific heat is not constant. §6 describes how the two basic theorems (H_1) and (H_2) of that paper follow from the present more direct and more general argument. The problem of reflection treated in Part II, §§7 and 8, is not of primary importance for a theoretical understanding of the mechanism of shocks, but is of great practial importance.

It is shown here that the algebra of that problem is entirely independent of the underlying thermodynamics (as is also abundantly clear from von Neumann's studies), and this algebra is brought into what I believe to be its most satisfactory mathematical form (by using $\sigma = \tan \delta_1 \cdot \tan \delta_2$ rather than $t = \tan \delta_1 / \tan \delta_2$ as the parameter which satisfies a quadratic equation and serves to determine the angles δ_1 and δ_2 of incidence and reflection in terms of the three states Z_1, Z_o, Z_2). One remarkable general proposition concerning the impossibility of glancing incidence in the limit for infinite pressure ratio is proved to hold under our wide thermodynamical assumptions if one adds one further quite natural hypothesis that is discussed in § 5. But all this is of little practical value as long as the transition from regular to Mach reflection is still a mystery, and to solve that mystery it seems wiser to follow von Neumann in limiting the investigation to the simplest cases (although no explanation is likely to be acceptable unless it carries over to the general situation here considered).

This memorandum is written without knowledge of H. A. Bethe's report, "The Theory of shock waves for an arbitrary equation of state", Division 8, NDRC, OSRD Report No. 545, to which I had no access.

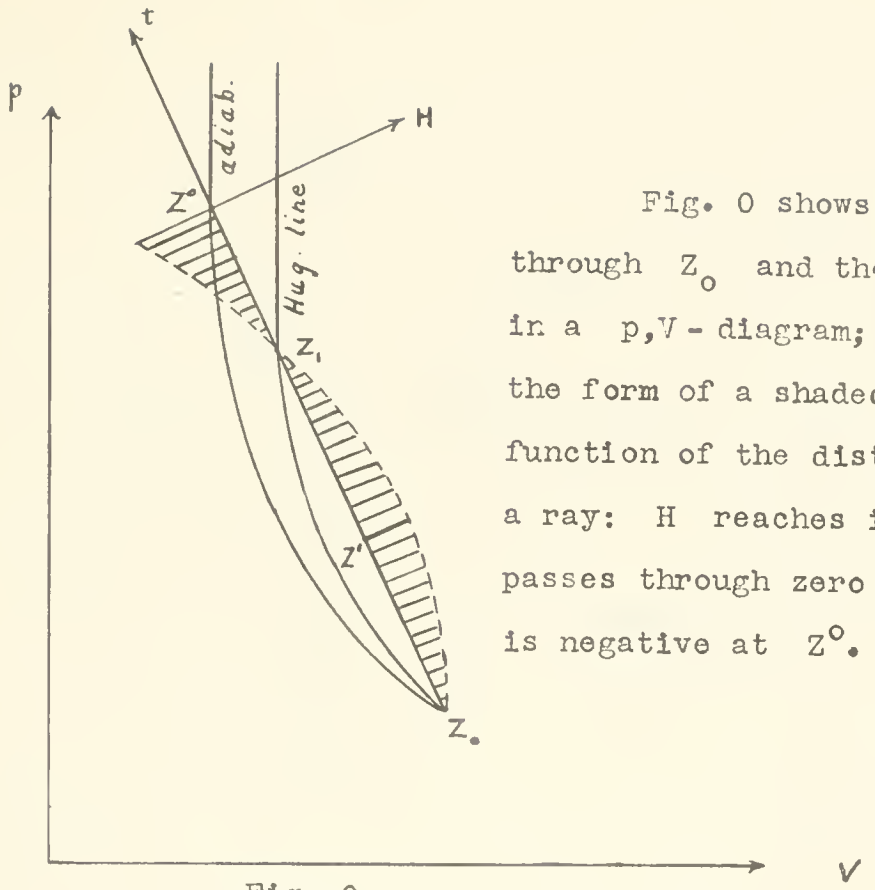

Fig. 0 shows the adiabatic through Z_0 and the Hugoniot line in a p,V - diagram; moreover, in the form of a shaded profile, H as function of the distance t along a ray: H reaches its maximum at Z' , passes through zero at Z_1 , and is negative at Z^0.

Fig. 0.

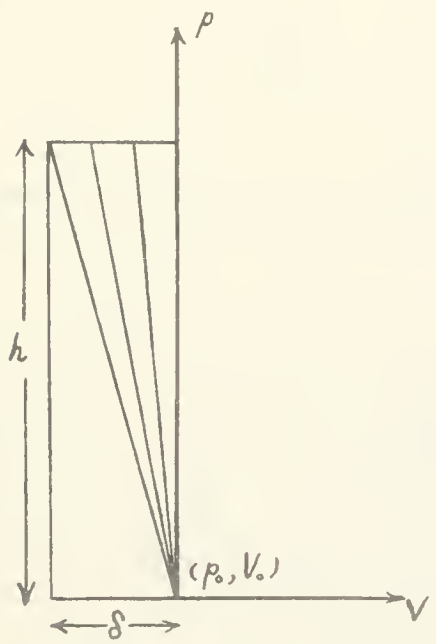

Fig. 1.

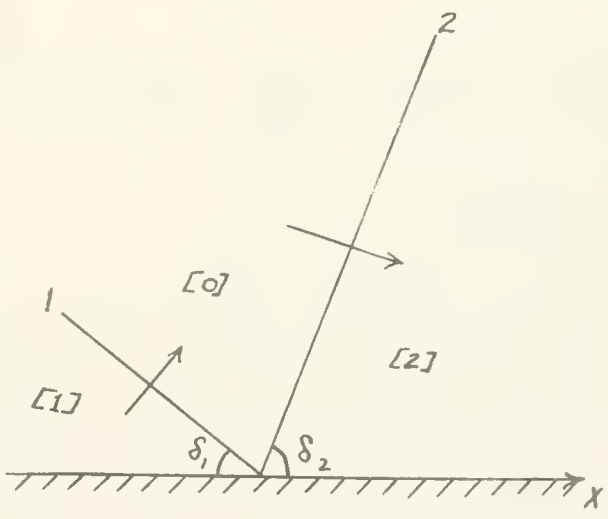

Fig. 2.

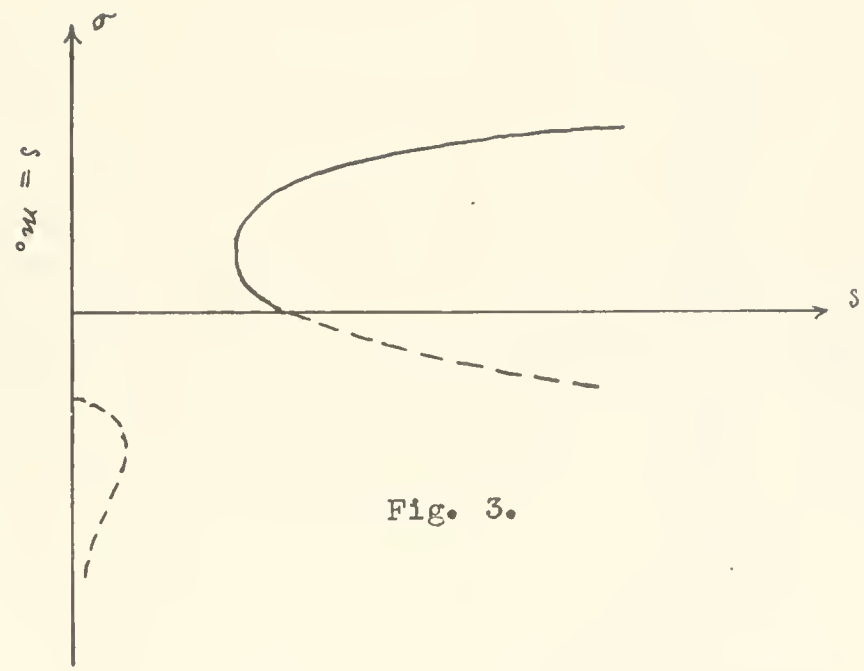

Fig. 3.

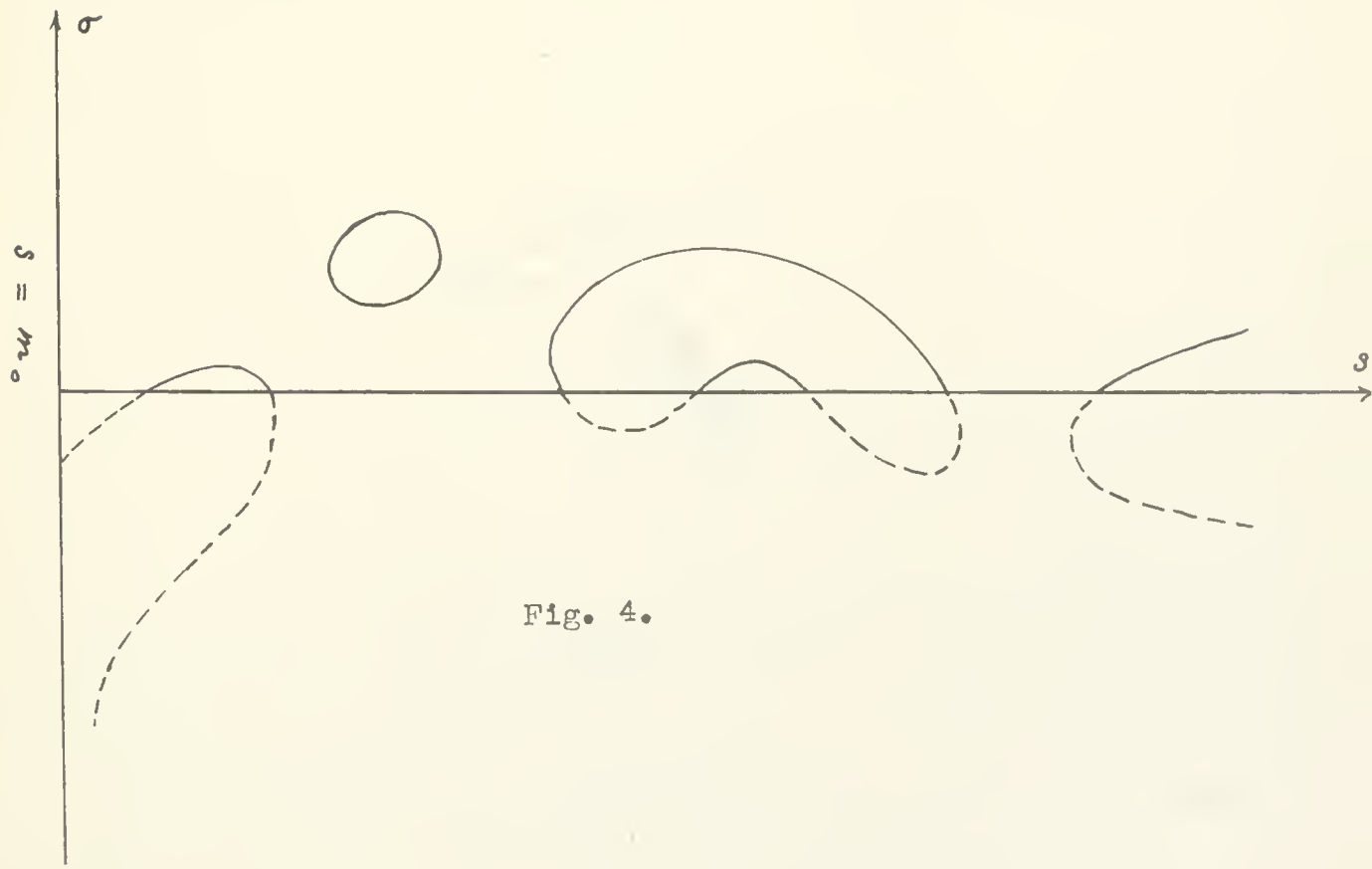

Fig. 4.

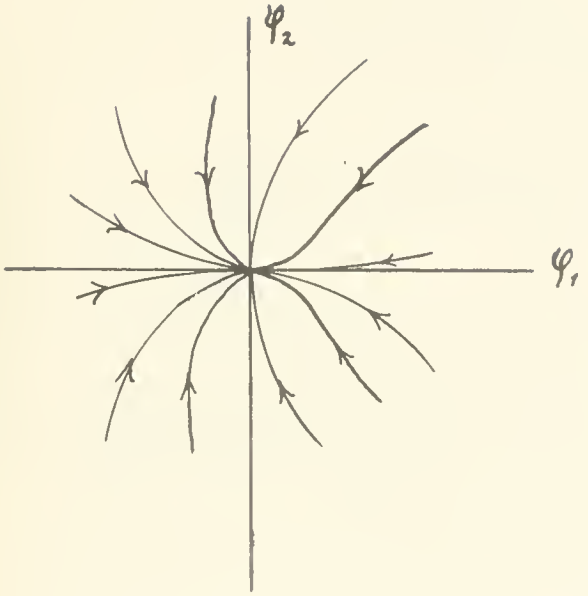

Fig. 5. Node.

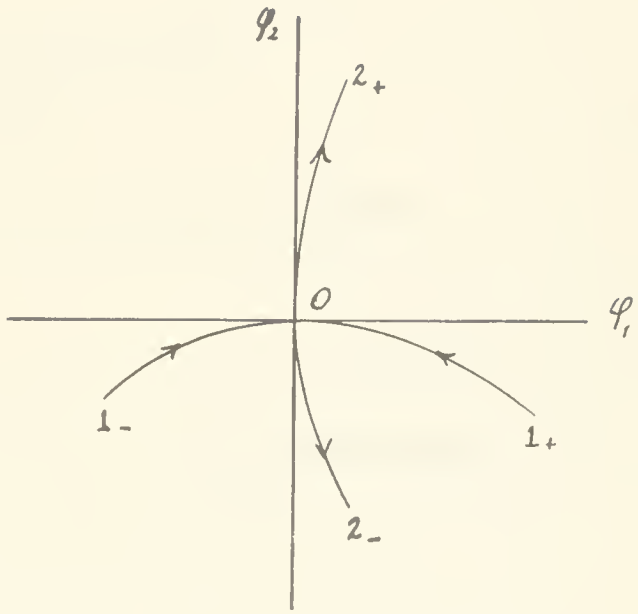

Fig. 6. Saddle

Fig. 7.

SHOCK WAVES IN ARBITRARY FLUIDS

by Hermann Weyl

Part I. Thermodynamics and the Shock Phenomenon

1. Shocks

At any place and time a fluid is in a definite thermodynamical state Z and in a definite state of motion, the latter being characterized by the components u, v, w of the velocity $\vec{u}$ in a Cartesian system of coordinates x, y, z. The possible thermodynamical states form a two-dimensional manifold $\mathfrak{Z}$; examples of quantities having definite values in a definite thermodynamical state are density ρ, pressure p, temperature T, volume, potential energy, heat content and entropy per mass unit, $V = \rho^{-1}$, E, $\mathfrak{i} = E + pV$, η. On $\mathfrak{Z}$ the fundamental thermodynamical equation

$$(1) \qquad dE = T\,d\eta - p\,dV$$

holds. It is once for all assumed that we are dealing with a stationary state $(Z, \vec{u})$; i.e., that neither Z nor $\vec{u}$ vary in time, but depend on x, y, z only.

In the limit for vanishing viscosity and heat conductivity, a discontinuity is possible such that $(Z, \vec{u})$ has a constant value $(Z_0, \vec{u}_0)$ for $x < 0$ and another constant value $(Z_1, \vec{u}_1)$ for $x > 0$. The symbols [0], [1] refer to the two regions $x < 0$, $x > 0$ which are divided by the shock front $x = 0$. Because of conservation

of mass, $\rho_o u_o = \rho_1 u_1$, and denoting by M this common flow of mass, we may write

$$u_o = M V_o, \quad u_1 = M V_1.$$

Then the laws of conservation of momentum and energy give the further relations

$$Mu_o + p_o = Mu_1 + p_1,$$

$$Mv_o = Mv_1, \quad Mw_o = Mw_1,$$

$$M\left\{i_o + \frac{1}{2}(u_o^2 + v_o^2 + w_o^2)\right\} = M\left\{i_1 + \frac{1}{2}(u_1^2 + v_1^2 + w_1^2)\right\}$$

whereas the law of increasing entropy ("More entropy flows in than out") requires

$$M\eta_1 \geqq M\eta_o$$

The phenomenon that results if $M \neq 0$ is called a shock. The problem then splits into two parts, one involving only the normal velocity component u, the other referring to the tangential components and stating that they go over the shock front unchanged:

(2) $$u_o = MV_o, \quad u_1 = MV_1,$$

(2') $$Mu_o + p_o = Mu_1 + p_1,$$

(2'') $$i_o + \frac{1}{2}u_o^2 = i_1 + \frac{1}{2}u_1^2$$

(3) $$v_o = v_1, \quad w_o = w_1$$

If, however, $M = 0$, we obtain a vortex sheet or slip stream characterized by the relations

$$u_o = u_1 = 0, \quad p_o = p_1 :$$

the fluid in [1] glides tangentially over that in [0] and has the same pressure on both sides. Our object will be

the study of shocks in an arbitrary fluid.

After introduction of the values (2) of the velocities (2') gives

$$M^2 = \frac{p_1 - p_0}{V_0 - V_1}, \tag{4}$$

and thereupon (2'') yields a relation between the two thermodynamical states Z_0 and Z_1; namely,

$$i_1 - i_0 = \frac{1}{2}M^2(V_0 - V_1)\cdot(V_0 + V_1) = \frac{1}{2}(p_1 - p_0)\,(V_0 + V_1) \text{ or}$$

$$E_1 - E_0 = \frac{1}{2}(p_1 + p_0)(V_0 - V_1)$$

Hence the problem of shocks is reduced to a study of this relation between two states

$$H \equiv H(Z_1, Z_0) \equiv (E_1 - E_0) - \frac{1}{2}(p_1 + p_0)(V_0 - V_1) = 0$$

(Hugoniot equation).

We are only interested in shocks in one and two dimensions. If in the latter case the normal unit-vector of the shock front in the direction [0] $\longrightarrow$ [1] is (α, β) instead of (1, 0) the formulas (2'), (3) for the velocities read as follows:

$$\alpha u_0 + \beta v_0 = MV_0, \quad \alpha u_1 + \beta v_1 = MV_1, \tag{5}$$

$$-\beta u_0 + \alpha v_0 = -\beta u_1 + \alpha v_1. \tag{5'}$$

A simple analysis (which will presently occupy us in much greater detail) reveals that the value of M resulting from (4) is of the order ρq where q is the acoustic velocity. Thus we are dealing with a situation like this: A certain quantity might be zero, but if it is not, it is even $\geqq 1$. Clearly quite different circum-

stances must be responsible for the two phenomena of shock and slip stream and M = 0 is in no way to be considered here the limiting case of a non-vanishing M. The problem of the shock layer will be compared to that of the slip layer in §9 by taking viscosity and heat conductivity into account and then letting them tend to zero. In particular, this passage to the limit will explain why a shock is not conservative with respect to entropy though it conserves energy.

2. Thermodynamical assumptions

Next we specify our assumptions concerning the thermodynamical behavior of our fluid. They will be enumerated, I-IV.

I. Infinitesimal adiabatic increase of pressure effects compression:

$$\left(\frac{dV}{dp}\right)_{ad} < 0.$$

II. The rate of compression - $\frac{dV}{dp}$ diminishes in this process:*

$$\left(\frac{d^2V}{dp^2}\right)_{ad} > 0.$$

These local hypotheses will be supplemented by two assumptions of "global" character.

III. In the continuous process of adiabatic compression one can raise pressure arbitrarily high.

* It can be made plausible that condition II is essential for the formation of a shock wave.

IV. The state Z is uniquely specified by pressure and specific volume, and the points (p,V) representing the possible states Z in a (p,V) - diagram form a convex region.

It would be more natural but less elementary to divide IV into a local and a global part, the local postulate asserting that in the neighborhood of a given state Z_o the variables p,V can be used as parameters for the specification of states; in other words that the projection $Z \rightarrow (p,V)$ of the manifold $\mathfrak{Z}$ of states Z upon a (p,V)-plane is locally one-to-one. The global part would assert that the projection of $\mathfrak{Z}$ covers a convex region $\bar{\mathfrak{Z}}$ in the (p,V)-plane and that Z_o being any given state and (p_o,V_o) its projection, one never runs against an obstacle when, starting at Z_o, one lets Z vary so that its projection (p,V) follows a given path in $\bar{\mathfrak{Z}}$ beginning at (p_o,V_o) ("continuation"). All this means that $\mathfrak{Z}$ is a covering surface over the convex $\bar{\mathfrak{Z}}$ without signularities and relative boundaries. But the convex $\bar{\mathfrak{Z}}$ being simply connected, this covering surface necessarily consists of a single sheet, or (p,V) determines Z uniquely. We thus fall back on the more elementary formulation IV. The assumption of convexity for the region $\bar{\mathfrak{Z}}$ will prove quite essential for our investigation.

Because of IV, entropy η is a (single-valued) function of p and V, $\eta = \eta(p,V)$. The adiabatic process of compression is thus defined by

$$\frac{\partial \eta}{\partial p}dp + \frac{\partial \eta}{\partial V}dV = 0$$

and since $\frac{\partial \eta}{\partial p}$, $\frac{\partial \eta}{\partial V}$ cannot vanish simultaneously, eq. (29), hypothesis I requires that $\frac{\partial \eta}{\partial p}$, $\frac{\partial \eta}{\partial V}$ are of the same sign:

$$\frac{\partial \eta}{\partial p} > 0, \frac{\partial \eta}{\partial V} > 0 \quad \text{or} \quad \frac{\partial \eta}{\partial p} < 0, \frac{\partial \eta}{\partial V} < 0.$$

As the region $\mathcal{J}$ is connected, the one set or the other will hold <u>everywhere</u>. We assume the first:

(I') $$\frac{\partial \eta}{\partial p} > 0, \frac{\partial \eta}{\partial V} > 0.$$

(The other alternative would make little difference.)

Infinitesimal adiabatic compression is now described by

$$dp = a^*.dt, \quad dV = b^*.dt; \quad a^* = \frac{\partial \eta}{\partial V}, \quad b^* = -\frac{\partial \eta}{\partial p}$$

(a <u>positive</u> dt corresponding to an <u>increase</u> of pressure).

Hypothesis II yields

$$\frac{d}{dt}\left(\frac{b^*}{a^*}\right) > 0 \quad \text{or} \quad b^*\frac{da^*}{dt} - a^*\frac{db^*}{dt} < 0,$$

i.e. $$b^*\left(\frac{\partial a^*}{\partial p}a^* + \frac{\partial a^*}{\partial V}b^*\right) - a^*\left(\frac{\partial b^*}{\partial p}a^* + \frac{\partial b^*}{\partial V}b^*\right) =$$

$$-a^{*2}\frac{\partial b^*}{\partial p} + a^*b^*\left(\frac{\partial a^*}{\partial p} - \frac{\partial b^*}{\partial V}\right) + b^{*2}\frac{\partial a^*}{\partial V} =$$

(II') $$\frac{\partial^2 \eta}{\partial p^2}\left(\frac{\partial \eta}{\partial V}\right)^2 - 2\frac{\partial^2 \eta}{\partial p \partial V}\frac{\partial \eta}{\partial V}\frac{\partial \eta}{\partial p} + \frac{\partial^2 \eta}{\partial V^2}\left(\frac{\partial \eta}{\partial p}\right)^2 < 0.$$

In the future we shall use hypotheses I and II in their analytic form (I') and (II').

Our next move consists in developing a number of consequences from the assumptions I-III. Let $(a,b) \neq (0,0)$ be two given numbers which determine the direction of a straight line through $Z_o = (p_o, V_o)$,

$$p = p_o + at, \quad V = V_o + bt,$$

the half-line or ray being obtained by the restriction $t \geq 0$

of the parameter t. We follow the straight line or the ray as long as it stays within $\mathcal{Z}$, and form

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial p}a + \frac{\partial \eta}{\partial V}b = \eta'$$

$$\frac{d\eta'}{dt} = \frac{\partial \eta'}{\partial p}a + \frac{\partial \eta'}{\partial V}b = \frac{\partial^2 \eta}{\partial p^2}a^2 + 2\frac{\partial^2 \eta}{\partial p \partial V}ab + \frac{\partial^2 \eta}{\partial V^2}b^2$$

According to (I'), η' is positive (negative) and hence η increasing (decreasing) along the whole straight line, provided $a \geqq 0$, $b \geqq 0$ ($a \leqq 0$, $b \leqq 0$).

Lemma 1. If $\eta' = 0$ for a certain value of t, then $\frac{d\eta'}{dt} < 0$ for the same value.

Proof. Substitute the values of a and b derived from $\eta' = 0$ into $\frac{d\eta'}{dt}$ and use the inequality (II').

Lemma 2. If $\eta' \leq 0$ for $t = 0$, then $\eta' < 0$ for $t > 0$.

Proof. First assume $\eta' < 0$ for $t = 0$. Should η change sign as Z travels along the ray, it would have to vanish somewhere. Suppose this occurs for the first time at $t = t_1$. As $\eta' < 0$ before t reaches this value, η' passes the zero level at t_1 ascending; hence $\frac{d\eta'}{dt} \geqq 0$ for $t = t_1$. But these two relations $\eta' = 0$, $\frac{d\eta'}{dt} \geqq 0$ for $t = t_1$, contradict Lemma 1.

If $\eta' = 0$ for $t = 0$ we have $\frac{d\eta'}{dt} < 0$ for $t = 0$: consequently η' becomes negative immediately after the point Z has started on its way, and from then on, as we have seen, η' must remain negative.

We turn to III. Starting at a given point $Z_o = (p_o, V_o)$ and raising p continually, we follow the adiabatic through Z_o; we thus obtain a continuous monotonely descending function V(p), and according to III we can make p travel over the entire infinite interval $p \geqq p_o$. While this happens, the directional coefficient $s = \frac{p - p_o}{V_o - V}$ of the straight line joing (p_o, V_o) with the point (p,V) on the adiabatic $\eta = \eta_o$ increases monotonely from a certain positive value m_o to $+\infty$. m denotes the adiabatic derivative $- \frac{dp}{dV}$ and m_o its value at the point Z_o. Consequently, there is exactly one point of the upper branch $p > p_o$ of the adiabatic on the ray from (p_o, V_o) of a given direction s, provided s lies between m_o and ∞.

The analytic proof for these intuitively evident statements runs as follows. Form the (non-total) differential

$$dr = (V_o - V)dp + (p - p_o)dV,$$

which vanishes along any ray through Z_o. We want to show that along the adiabatic dr is positive for a positive increment dp:

$$R = (V_o - V) + (p - p_o)\frac{dV}{dp} > 0.$$

By hypothesis II

$$(6) \quad \frac{dR}{dp} = (p - p_o)\frac{d^2V}{dp^2} > 0 \qquad \text{for } p > p_o,$$

and since R vanishes for $p = p_o$ it must indeed be positive for $p > p_o$.

(7) $$R = \int_{p_o}^{p} (p - p_o) \frac{d^2V}{dp^2} \, dp > 0.$$

We may write

(8) $$dr = (V_o - V)^2 . ds$$

and thus the monotone behavior of s is exhibited by the equation

$$\frac{ds}{dp} = R/(V_o - V)^2$$

It is clear that s tends to m_o for $p \to p_o$, and combination of hypothesis III with the trivial inequality $s > (p - p_o)/V_o$ leads to $s \to \infty$ for $p \to \infty$.

3. The fundamental inequality for the direction $\overrightarrow{Z_o Z_1}$

For the differential of

$$H(ZZ_o) = (E - E_o) - \tfrac{1}{2}(p + p_o)(V_o - V)$$

we find by (1)

$$dH = dE - \tfrac{1}{2}(V_o - V)dp + \tfrac{1}{2}(p_o + p)dV = Td\eta - \tfrac{1}{2}\{(V_o - V)dp + (p - p_o)dV\},$$

(9) $$dH = Td\eta - \tfrac{1}{2} dr,$$

and hence along any ray

(10) $$dH = Td\eta .$$

Let Z_o, Z_1 be two distinct states satisfying the Hugoniot equation $H(Z_1, Z_o) = 0$. Making use of the convexity of the region $\mathcal{J}$ (hypothesis IV) we join the two representing points $(p_o V_o)$, $(p_1 V_1)$ in the (p V) -diagram by a straight segment. Integrating (10) along it we obtain

(11) $$H(Z_1 Z_o) = \int_{Z_o}^{Z_1} Td\eta = 0.$$

From this simple relation we are going to deduce

Theorem 1. For any two distinct states Z_o, Z_1 linked by Hugoniot's equation $H = 0$ the following inequalities hold

(12) $N^* = (p_1 - p_o)(V_o - V_1) > 0,$

(13) $(p_1 - p_o) + m_o(V_1 - V_o) > 0, \quad (p_1 - p_o) + m_1(V_1 - V_o) < 0$

Before proving the theorem let us discuss its physical significance. We have seen that elimination of the velocities from the conditions for a shock lead to the Hugoniot equation and the inequality $N^* \geq 0$. We now realize that we may omit the supplementary relation $N^* \geq 0$ because it follows, even in the sharper form $N^* > 0$, from the Hugoniot equation.

The adiabatic derivative

$$\frac{dp}{d\rho} = -V^2 \cdot \frac{dp}{dV} = V^2 \cdot m$$

is the square of the "acoustic velocity" q. Assume $V_o > V_1$. Then the two relations (13) give

(14) $$m_o < M^2 = \frac{p_1 - p_o}{V_o - V_1} < m_1$$

or $\rho_o q_o < |M| < \rho_1 q_1$, thus confirming a statement made in §1 on the magnitude of M. In terms of the velocities u_o, u_1 given by

$$u_c^2 = \frac{p_1 - p_o}{V_o - V_1} \cdot V_o^2 \qquad u_1^2 = \frac{p_1 - p_o}{V_o - V_1} \cdot V_1^2$$

(14) may be written as

(15) $|u_o| > q_o, \quad |u_1| < q_1$:

relatively to the shock front the flow in [0] is supersonic, in [1] subsonic.

Proceeding to the proof of our theorem, let us travel from Z_0 along that ray which passes through Z_1 and hence set in the two lemmas

$$a = p_1 - p_0, \quad b = V_1 - V_0.$$

Were $a \geqq 0$, $b \geqq 0$, then η' would be positive and hence η monotone increasing along the segment Z_0Z_1, which contradicts the relation (11). Consequently this combination as well as the other $a \leqq 0$, $b \leqq 0$ are ruled out, and therefore $ab < 0$, or (12) must hold.

Next consider

$$\eta' = a\frac{\partial \eta}{\partial p} + b\frac{\partial \eta}{\partial V}$$

Were $\eta' \leqq 0$ for $t = 0$, then η' would be negative and hence η would monotonely decrease while Z travels along the straight segment from Z_0 to Z_1, in contradiction to the equation (11). Therefore

$$a\left(\frac{\partial \eta}{\partial p}\right)_0 + b\left(\frac{\partial \eta}{\partial V}\right)_0 > 0,$$

and this is identical with the first of the inequalities (13), from which the second follows by interchanging Z_0 and Z_1.

But our argument shows much more. While Z travels along the ray from Z_0 passing through Z_1, η' starts with a positive value; because of (11) it must change sign before Z reaches Z_1. But η' remains negative from the moment it vanishes for the first time. Rise and fall of $H = H(ZZ_0)$ along the ray is coupled with that of η by the relation

$dH = Td\eta$. Hence H first rises monotonely to a positive maximum and then decreases, on the descent passing through 0 for $Z = Z_1$. This description shows that H is positive from Z_o to Z_1 and negative after. In particular:

On the ray from Z_o through Z_1 the point Z_1 is the only one which satisfies the Hugoniot equation $H = 0$.

According to (12) either the inequalities $p_1 > p_o$, $V_o > V_1$ or $p_o > p_1$, $V_1 > V_o$ hold for two distinct states Z_o, Z_1, satisfying the Hugoniot equation $H(Z_1, Z_o) = 0$. We shall indicate these alternatives by $Z_1 > Z_o$, $Z_1 < Z_o$ respectively. By the Hugoniot contour $\mathcal{H}$ (for a given Z_o) we understand the locus of all points $Z_1 = (p_1 V_1)$ for which

$$H(Z_1 Z_o) = 0 \quad \text{and} \quad Z_1 > Z_o.$$

(It is quite essential for our argument to pick this upper branch $Z_1 > Z_o$.) Our above result may then be stated thus: On the Hugoniot contour $s = \frac{p_1 - p_o}{V_o - V_1}$ lies between m_o and ∞ ; s is a uniformizing parameter inasmuch as the value of s specifies uniquely the point Z_1. But it must be borne in mind that so far we have not yet proved that to every preassigned value of $s > m_o$ there actually corresponds a point on the Hugoniot contour; we only know that there cannot be more than one.

4. Entropy and parametrization of the Hugoniot line

For a moment let us return to the upper branch $\mathcal{A}$ of the adiabatic through Z_o, on which p increases monotonely from p_o to $+\infty$ and the directional parameter $s = \frac{p - p_o}{V_o - V}$ from m_o

to ∞. Moving along it we have by (9) $dH = -\frac{1}{2}dr = -\frac{1}{2}R.dp$ and hence (7) implies

Lemma 3. $H = H(ZZ_o)$ is negative along $\mathcal{H}$.

As a matter of fact, we even know that H is falling, $R > 0$, and falling with increasing rapidity, $\frac{dR}{dp} > 0$, eq. (6). The explicit formula giving $H(Z^oZ_o)$ for any point Z^o on $\mathcal{H}$ is

$$H(Z^oZ_o) = -\frac{1}{2}\int_{p_o}^{p^o} Rdp = -\frac{1}{2}\int_{p_o}^{p^o}(p - p_o)(p^o - p)\frac{d^2V}{dp^2}dp < 0.$$

This lemma is instrumental in establishing the following two propositions:

Theorem 2. For any two distinct states Z_o, Z_1 satisfying the Hugoniot equation one has $\eta > \eta_o$ or $\eta_1 < \eta_o$ according to whether $Z_1 > Z_o$ or $Z_1 < Z_o$.

Theorem 3. The Hugoniot contour is a simple line starting from Z_o on which s and η are monotone increasing (s traveling from m_o to $+\infty$, η from η_o to an unknown destination).

Proof. Assume the arrangement $Z_1 > Z_o$ for the two points Z_o, Z_1 linked by the Hugoniot equation. Draw the ray from Z_o passing through Z_1: its direction $s = \frac{p_1 - p_o}{V_o - V_1}$ lies between m_o and $+\infty$, and hence the ray meets the upper branch of the adiabatic line $\mathcal{H}$ in a point $Z^o = (p^o, V^o)$ where, according to Lemma 3, $H = \int_{Z_o}^{Z^o} Td\eta$ is negative. Hence η on its road along the ray must have passed the point Z' where it reaches its maximum, and running down the hill have passed a point Z_1 for which $H = \int_{Z_o}^{Z_1} Td\eta = 0$ before coming

to Z^o. Its value η_1 at the point Z_1 of the Hugoniot contour is therefore higher than its value η_o at Z^o (Theorem 2).

Again, let s be any value between m_o and ∞. The ray going out from Z_o in the direction s meets $\mathcal{H}$ in a point Z^o where H is negative. Hence H must vanish at a certain point Z_1 between Z' and Z^o (where η is already on the down grade), and thus a point on the Hugoniot contour is obtained corresponding to this preassigned value of $s = \frac{p_1 - p_o}{V_o - V_1}$. We know it is the only one. The question raised at the end of § 3 is answered in the affirmative, and we may now speak of the Hugoniot contour as a simple line $Z_1 = \phi(s)$ with the uniformizing parameter s.

Because the point $Z_1 = \phi(s)$ is thought of as varying along the Hugoniot line, let us drop the index 1. Along $\mathcal{H}$ we may use the differential relation dH = 0 or, according to (9) and (8),

$$Td\eta - \frac{1}{2}dr = Td\eta - \frac{1}{2}(V_o - V)^2 ds = 0.$$

Because

$$\frac{d\eta}{ds} = \frac{(V_o - V)^2}{2T}$$

thus turns out to be positive, η increases all the way along the Hugoniot line with s. This gives a new proof for the inequality $\eta_1 > \eta_o$ by integration along that line,

$$\eta_1 - \eta_o = \int_{m_o}^{s_1} \frac{(V_o - V)^2}{2T} ds \qquad (s_1 > m_o)$$

Thus Theorem 3 is proved, and Theorem 2 even by two different methods.

5. The limit of pressure along the Hugoniot line

The straight segment Z_0Z^0 of direction $s(> n_0)$ joining Z_0 with a point Z^0 on $\mathcal{C}$ converges toward the vertical ray $V = V_0$, $p \geq p_0$ if s tends to infinity. It is therefore natural to make the additional hypothesis

V. *If* (p_0, V_0) *lies in* $\mathcal{Z}$ *so does the entire vertical ray* $V = V_0$, $p_0 \leq p < \infty$.

Physically this means that when the unit quantum of fluid in the state (p_0, V_0) is sealed up in a vessel of volume V_0 and then heated, its pressure can be raised to arbitrarily high values. I maintain:

Theorem 4. Under the additional hypothesis V *the function* p(s) *representing pressure as function of* s *along the Hugoniot line, tends to infinity with* $s \to \infty$.

p tends to infinity along the adiabatic line (hypothesis III) as well as along the vertical $V = V_0$ (hypothesis V); the same behavior along the Hugoniot line will be shown to be a consequence of these two circumstances. We argue as follows.

Let h be an arbitrarily large positive constant. The section $p_0 \leq p \leq p_0 + h$ of the vertical $V = V_0$ lies in $\mathcal{Z}$ and hence the same is true for a certain strip S of small width δ,

$$V_0 - \delta \leq V \leq V_0, \quad p_0 \leq p \leq p_0 + h$$

to the left of it. We may ascertain two positive constants

c, C such that $\frac{\partial \eta}{\partial p} \geq c$, $\frac{\partial \eta}{\partial V} \leq C$ in S. Consider the section $0 \leq t \leq h$ of the ray $p = p_o + t$, $V = V_o - \lambda t$. If the positive constant λ which determines its direction is less than $\frac{\delta}{h}$ that section lies in S and therefore

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial p} - \lambda \frac{\partial \eta}{\partial V} > 0$$

provided $\lambda < c/C$. Previously we have characterized the direction of such a ray by the number $s = 1/\lambda$. Hence η is on the ascent along the whole segment $p_o \leq p \leq p_o + h$ of a ray of direction s provided $s > \omega = \max(h/\delta, C/c)$. Such a ray will meet the Hugoniot line in a point $\Phi(s) = (p(s), V(s))$ <u>after</u> η has begun to decrease, and hence the corresponding value p(s) is necessarily larger than $p_o + h$. In other words: Given any $h > 0$ we may ascertain a number ω such that $p(s) > p_o + h$ for $s > w$. The statement of Theorem 4 is the customary abbreviation for this fact.

6. The example of the ideal gas

An ideal gas is characterized by the equation of state $pV = RT$. We normalize units so as to make the constant $R = 1$. The energy is a monotone increasing function of temperature alone, $E = f(T)$, and

$$\eta = \int \frac{f'(T)}{T} dT + \log V.$$

One easily computes

$$-\left(\frac{dp}{dV}\right)_{ad} = \frac{p}{V}\cdot\frac{f'+1}{f'} ,$$

$$\left(\frac{d^2p}{dV^2}\right)_{ad} = \frac{p}{V^2}\cdot\frac{Wf}{f'^3}$$

where f', f" stand for $\frac{df}{dT}$, $\frac{d^2f}{dT^2}$ and

$$Wf = f'(f' + 1)\ (2f' + 1) - Tf".$$

Condition I is satisfied, II requires

$$Wf > 0.$$

Hypotheses IV and V hold, the convex region $\mathcal{Z}$ being the quadrant $p > 0$, $V > 0$. The adiabatic line through Z_o,

$$\int_{T_o}^{T} \frac{f'(T)}{T}dT + \log \frac{V}{V_o} = 0,$$

is best parametrized in terms of T: while T increases from T_o to infinity, V diminishes according to the law

$$\log V/V_o = -\int_{T_o}^{T} (f'/T)dT,$$

and hence $p = T/V$ tends to infinity even faster than T ($p \geqq T/V_o$); thus statement III is verified. (It ought to be observed that V is not sure to tend to 0 in this process; if the integral $\int_{T_o}^{\infty} (f'/T)dT$ converges, there will be a positive limiting value. Nor is it sure that V tends to ∞ in the opposite direction. Thus there is good reason for using the upper branch of the adiabatic and for not going beyond the statement III.)

In a previous note I found that for an ideal gas the Hugoniot equation is solvable by a simple extraction of square roots if one uses T as the parameter of the Hugoniot line: change the rectangle with the Sides T_o, T into one of equal area the sides ϑ_o, ϑ of which have the increased difference

$$(T - T_o) + 2(f(T) - f(T_o))$$

Then the point Z = (p,V) determined by

$$V/V_o = \vartheta_o/T_o, \quad V_o/V = \vartheta/T; \quad p_o/p = \vartheta_o/T, \quad p/p_o = \vartheta/T_o$$

describes the Hugoniot line. The corresponding values of η and s are

$$\eta - \eta_o = \int_{T_o}^{T} (f'/T)dT + \log(\vartheta_o/T_o),$$

$$s = \frac{p - p_o}{V_o - V} = \frac{p_o}{V_o} \cdot \frac{\vartheta - T_o}{T_o - \vartheta} = \frac{p}{V} \cdot \frac{T - \vartheta_o}{\vartheta - T}$$

Consequently our investigation yields the following mathematical theorem:

The assumptions $f' > 0$, $Wf > 0$ imply

$$(16) \qquad \frac{\vartheta - T_o}{T_o - \vartheta_o} > \frac{f'(T_o) + 1}{f'(T_o)}, \quad \frac{T - \vartheta_o}{\vartheta - T} < \frac{f'(T) + 1}{f'(T)};$$

$$(17) \qquad \eta - \eta_o > 0.$$

While T thus appears as a monotone parameter for the Hugoniot line we know from our general theory (Theorem 3) that η is of the same character; hence $\frac{d\eta}{dT}$ ought to be positive throughout. Indeed, a simple calculation gives the expression

$$\frac{d\eta}{dT} = \frac{f'(T)(\vartheta + \vartheta_0 - 2T) + (\vartheta - T)}{T(\vartheta + \vartheta_0)} = \varphi(T)$$

the numerator of which is positive according to the second of the inequalities (16). Hence (17) is an elementary consequence of (16),

$$\eta - \eta_0 = \int_{T_0}^{T} \varphi(T)\,dT.$$

However, it remains a challenge to the mathematician to find for the theorem (16) a simple direct proof that is not a mere disguise of the "shock thermodynamics" of an ideal gas.

Part II. Reflection of a Shock Wave

7. The algebra of reflection

We study a two-dimensional fluid covering the semi-plane $y > 0$ bounded by the wall of the x-axis and two oblique shock fronts meeting at the origin. Notations are indicated by Figure 2. Let $(\alpha_1 \beta_1)$, $(\alpha_2 \beta_2)$ be the

normals in the directions [1] → [0], [0] → [2] respectively, and $(u_1,0),(u_o,v_o),(u_2,0)$ be the velocities. Continuity of the tangential velocity component across the shock front 1, eq. (5'), gives

$$\beta_1 u_1 = \beta_1 u_o - \alpha_1 v_o \text{ or } u_1 = \frac{1}{\beta_1}(\beta_1 u_o - \alpha_1 v_o),$$

and thus (5) changes into

$$(A_1)\ \alpha_1 u_o + \beta_1 v_o = M_1 V_o,\quad (A'_1)\ \frac{\alpha_1}{\beta_1}(\beta_1 u_o - \alpha_1 v_o) = M_1 V_1$$

while M_1 satisfies the equation

$$M_1^2 = (p_1 - p_o)/(V_o - V_1)$$

Similar relations prevail for the second front:

$$(A_2)\ \alpha_2 u_o + \beta_2 v_o = M_2 V_o,\quad (A'_2)\ \frac{\alpha_2}{\beta_2}(\beta_2 u_o - \alpha_2 v_o) = M_2 V_2.$$

We may eliminate the velocity components u_o, v_o and obtain two relations between the two directions, either by equating the values of u_o, v_o derived from (A_1, A'_1) respectively:

$$(18_u)\quad \frac{M_1}{\alpha_1}(\alpha_1^2 V_o + \beta_1^2 V_1) = \frac{M_2}{\alpha_2}(\alpha_2^2 V_o + \beta_2^2 V_2)\ (= u_o),$$

$$(18_v)\quad M_1 \beta_1 (V_o - V_1) = M_2 \beta_2 (V_o - V_2)\ (= v_o),$$

or, by equating the values of u_o, v_o derived from (A_1, A_2) and from (A'_1, A'_2) respectively:

(19_u)

$$(19_v)\quad V_o (M_1 \alpha_2 - M_2 \alpha_1) = \frac{\beta_1 \beta_2}{\alpha_1 \alpha_2}(V_1 M_1 \alpha_2 - V_2 M_2 \alpha_1).$$

(The first equation of this equivalent second set is omitted because it will not be needed.)

We discuss the signs by means of the first elimination. For geometric reasons α_1, α_2, $\alpha_2 \beta_1 - \alpha_1 \beta_2$ are positive; hence by (18_u) M_1, M_2 are of equal sign, and since the regions [1] and [2] get interchanged and M_1, M_2 replaced by $-M_2$, $-M_1$ if the direction of the x-axis is reversed, we may assume $M_1 > 0$, $M_2 > 0$. The law of increasing entropy requires

$$M_1(\eta_o - \eta_1) \geqq 0, \quad M_2(\eta_2 - \eta_o) \geqq 0,$$

which gives $\eta_1 \leqq \eta_o \leqq \eta_2$, and thus by Theorem 2 the arrangement $Z_1 < Z_o < Z_2$ must prevail,

(20) $$p_1 < p_o < p_2, \quad V_1 > V_o > V_2$$

Then (18_v) shows β_1 and β_2 to be of opposite signs, and thus $\beta_1 > 0$, $\beta_2 < 0$ (because $\beta_1 < 0$, $\beta_2 > 0$ would contradict $\alpha_2 \beta_1 - \alpha_1 \beta_2 > 0$).

We introduce the abbreviations

$$p_o - p_1 = p_1^*, \quad p_2 - p_o = p_2^*; \quad V_1 - V_o = V_1^*, \quad V_o - V_2 = V_2^*$$

so that $M_1^2 = p_1^*/V_1^*$, $M_2^2 = p_2^*/V_2^*$.

Assuming three states Z_1, Z_o, Z_2 to be given in agreement with the inequalities (20) let us try to solve in the most convenient fashion the algebraic equations that

determine the normals $(\alpha_1\ \beta_1)$, $(\alpha_2\ \beta_2)$ of the two shock fronts. The double expression for u_o/v_o derived from (18_u), (18_v) gives

(21) $(\alpha_1^2 V_o + \beta_1^2 V_1)\ \alpha_2\ \beta_2(V_o - V_2) =$

$$(\alpha_2^2 V_o + \beta_2^2 V_2)\ \alpha_1\ \beta_1(V_o - V_1).$$

This is a bilinear homogeneous relation with respect to the two sets of variables $t_1 : t_2 = \alpha_1\beta_2 : \alpha_2\beta_1$ and $\sigma_1 : \sigma_2 = \alpha_1\alpha_2 : \beta_1\beta_2$,

(22) $A_{11}t_1\sigma_1 + A_{12}t_1\sigma_2 + A_{21}t_2\sigma_1 + A_{22}t_2\sigma_2 = 0$,

with the coefficients

$$A_{11} = V_o V_2^*,\quad A_{12} = V_2 V_1^*,\quad A_{21} = V_o V_1^*,\quad A_{22} = V_1 V_2^*.$$

Write (19_v) or $\sigma_1 V_o(M_1\alpha_2 - M_2\alpha_1) = \sigma_2(V_1 M_1\alpha_2 - V_2 M_2\alpha_1)$ in the form

$$\alpha_1 M_2(V_o\sigma_1 - V_2\sigma_2) = \alpha_2 M_1(V_o\sigma_1 - V_1\sigma_2)$$

and multiply at left and right by $M_2(V_o - V_2)\beta_2 = M_1(V_o - V_1)\beta_1$ respectively. The result is another bilinear relation

(23) $B_{11}t_1\sigma_1 + B_{12}t_1\sigma_2 + B_{21}t_2\sigma_1 + B_{22}t_2\sigma_2 = 0$

with the coefficients

$$B_{11} = V_o p_2^*,\quad B_{12} = -V_2 p_2^*,\quad B_{21} = V_o p_1^*,\quad B_{22} = -V_1 p_1^*.$$

One may look upon (22) and (23) as two linear homogeneous equations for σ_1, σ_2 by the elimination of which one gets a quadratic equation for $t_1 : t_2$; after solving it either (22) or (23) gives $\sigma_1 : \sigma_2$ as a broken linear function of t_1, t_2. One may also work the scheme, as we shall do here,

the other way around, by first eliminating t_1, t_2, thus obtaining the formulas

$$\begin{vmatrix} A_{11}\sigma_1 + A_{12}\sigma_2, & A_{21}\sigma_1 + A_{22}\sigma_2 \\ B_{11}\sigma_1 + B_{12}\sigma_2, & B_{21}\sigma_1 + B_{22}\sigma_2 \end{vmatrix} = 0,$$

$$-\frac{t_1}{t_2} = \frac{A_{11}\sigma_1 + A_{12}\sigma_2}{A_{21}\sigma_1 + A_{22}\sigma_2} = \frac{B_{11}\sigma_1 + B_{12}\sigma_2}{B_{21}\sigma_1 + B_{22}\sigma_2}.$$

The quadratic equation for $\sigma_1 : \sigma_2$,

$$L_{11}(V_o\sigma_1)^2 + L_{12}V_o\sigma_1\sigma_2 + L_{22}\sigma_2^2 = 0 \tag{24}$$

is readily seen to have the coefficients

$$L_{11} = V_1^* p_2^* - V_2^* p_1^*, \quad L_{22} = -V_1V_2(V_2^* p_2^* - V_1^* p_1^*),$$
$$L_{12} = (p_1^* + p_2^*)(V_1V_2^* - V_2V_1^*).$$

We introduce the non-homogeneous variables $\sigma = -V_o\sigma_1/\sigma_2$, $t = -t_1/t_2$, for which ∞ is a permissible value, and thus arrive at the final equations

$$L_{11}\sigma^2 - L_{12}\sigma + L_{22} = 0,$$

$$t = \frac{V_1V_2^* - V_1^*\sigma}{V_2V_1^* - V_2^*\sigma} = \frac{p_1^*(V_1 + \sigma)}{p_2^*(V_2 + \sigma)}$$

(21) shows that $\alpha_1\beta_1$ and $\alpha_2\beta_2$ must have opposite signs, $\alpha_1\beta_1 \cdot \alpha_2\beta_2 = \alpha_1\alpha_2 \cdot \beta_1\beta_2 \leqq 0$ or $\sigma \geqq 0$. Vice versa, if this condition is satisfied, the second expression for t proves this quantity to be positive, and hence the square roots of

$$V_o\left(\frac{\alpha_1}{\beta_1}\right)^2 = \sigma t, \quad V_o\left(\frac{\alpha_2}{\beta_2}\right)^2 = \sigma t^{-1}$$

may be extracted:

$\alpha_1/\beta_1 = \tan\delta_1 = \sqrt{(\sigma t/V_o)}$; $-\alpha_2/\beta_2 = \tan\delta_2 = \sqrt{(\sigma/tV_o)}$.

Let the two states $Z_1 < Z_o$ be given: the possible states $Z_2 > Z_o$ related to Z_o by the Hugoniot equation may be specified by the directional parameter $s = p_2^*/V_2^*$. The relation (25) then becomes an equation $F(s,\sigma) = 0$ which defines a contour f in a (s,σ) -diagram. Since all quantities, including the normals of the shock fronts, are uniquely expressed in terms of the variables s and σ the situation is best discussed by means of this contour.

8. Why glancing incidence is impossible

We now reintroduce the thermodynamical assumptions I-IV of §2, and the fact that Z_1 and Z_o as well as Z_o and Z_2 are linked by the Hugoniot equation. We then learn from Theorem 1 or eq. (14) that

(27) $m_1 < p_1^*/V_1^* < m_o < p_2^*/V_2^* < m_2$,

consequently $p_1^*/V_1^* < p_2^*/V_2^*$ or $L_{11} = V_1^* p_2^* - V_2^* p_1^* > 0$. Thus $\sigma_1 : \sigma_2 = 1 : 0$ or $\sigma = \infty$ cannot be a solution of (24), hence glancing incidence is impossible. $V_o^2(p_1^*/V_1^*)$, $V_o^2(p_2^*/V_2^*)$ are the squares $|u_o^{(1)}|^2$, $|u_o^{(2)}|^2$, of the components of the velocity $\vec{u_o}$ in the direction of the normals 1 and 2. Thus the inner part of the inequalities (27) may be written as

$$|u_o^{(1)}| < q_o < |u_o^{(2)}|,$$

and the prevention of glancing incidence is seen to be due to the separation of the two normal components $|u_o^{(1)}|$, $|u_o^{(2)}|$

of the velocity in the middle region [0] by the acoustic velocity q_o prevailing there: the stationary flow in [0] is subsonic as far as the component $|u_o^{(1)}|$ in the direction of the first normal is concerned; it is supersonic with respect to $|u_o^{(2)}|$.

We could have demonstrated the impossibility of glancing incidence more directly as follows. The equation (18_u) shows that α_1, α_2 may vanish only together, or that $\alpha_1 \alpha_2 = 0$ implies the vanishing of both α_1 and α_2 (head-on impact). In that case $\beta_1^2 = \beta_2^2 = 1$, and the second equation (18_v) yields $M_1^2(V_1 - V_o)^2 = M_2^2(V_o - V_2)^2$ or $p_1^* V_1^* = p_2^* V_2^*$. Likewise the relation (18_v) proves that $\beta_1 \beta_2 = 0$ implies the vanishing of both β_1 and β_2 (glancing incidence) and in that case we must have $\alpha_1^2 = \alpha_2^2 = 1$ and thus $M_1^2 = M_2^2$ or $p_1^*/V_1^* = p_2^*/V_2^*$ by (18_u). It is the latter relation which is made impossible by our fundamental inequality (27). (By the way, these considerations reveal that the equation (24) holds even in the two limiting cases of head-on impact and glancing incidence, although the conditions $\alpha_1 \beta_2 = \alpha_2 \beta_1 = 0$ then prevailing invalidate its derivation.

I entered into a discussion of the problem of reflection mainly for the purpose of showing that the inequalities (15) for a linear shock are responsible for the simple fact which I once more repeat as

Theorem 5. Under the hypotheses enumerated at I-IV in §2, glancing incidence is impossible.

For the discussion of the contour $\mathcal{f}$ in the (s, σ) -diagram it is convenient first to ignore the condition $\sigma \geqq 0$. The contour is such that to every abscissa $s > m_o$ there correspond no or two values of the ordinate σ. As long as we keep away from the ends $s \to m_o$ and $s \to \infty$, σ will keep within finite bounds. Hence, generally speaking, the contour will consist of a number of separate lines,* all closed with the possible exception of one opening toward $s \to m_o$ and one opening toward $s \to \infty$. Fig. 3 shows the actual situation for an ideal gas, Fig. 4 a more complicated possibility such as may occur for a fluid of fairly wild thermodynamical behavior. The horizontal $\sigma = 0$ cuts away the "submerged" part $\sigma < 0$ which is without real significance, and in the figure is indicated by a broken line. The picture makes it quite clear that head-on impact is a perfectly normal and regular occurrence. Whereas the details elude general analysis, we can deduce from (25) fairly accurately the behavior of the contour toward the two ends $s \to m_o$ and $s \to \infty$.

p_1, V_1, p_o, V_o are fixed, while p_2, V_2 vary on the contour; but it must be borne in mind that V_2 never exceeds V_o. With s decreasing to m_o the quantities p_2^*, V_2^* converge to zero, and thus the equation (25) reveals that from a certain moment on there are two real roots σ tending

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* I use the word line for the trace of a continuous motion (= continuous image of an interval), contour for the locus of points in a plane satisfying an equation (e.g. the isohypses on a contour map).

to $-V_1$ and $-\infty$ respectively. Hence for s sufficiently close to its lower limit m_o, the contour $\mathcal{C}$ has two branches, but both are in the submerged region and therefore without real significance.

Much more interesting is the other limit $s \to \infty$. Here the hypothesis V (§5) and its consequence, Theorem 4,

$$p_2^* \to \infty \text{ for } s \to \infty,$$

come into play. Divide the equation (25) by p_2^*. This done, one sees that for sufficiently high s its coefficients deviate as little as one wishes from

$$\overline{L_{11}} = V_1^*, \quad \overline{L_{22}} = -V_1V_2V_2^*, \quad \overline{L_{12}} = V_1V_2^* - V_2V_1^*.$$

The discriminant of the $\overline{L}$ equals

$$\overline{L_{12}}^2 - 4\overline{L_{11}}\overline{L_{22}} = (V_1V_2^* + V_2V_1^*)^2 = V_o^2(V_1 - V_2)^2 \geqq V_o^2(V_1 - V_o)^2.$$

Hence as s approaches infinity we shall have, from a certain moment on, two distinct real roots σ', σ'' or branches of the contour. Let σ' refer to the upper, σ'' to the lower branch, $\sigma' > \sigma''$, and use the notation $f(s) \sim g(s)$ as an abbreviation for $\lim_{s \to \infty} (f(s) - g(s)) = 0$. We then find the following limit relations

$$\sigma' \sim -V_1 \cdot \frac{V_o - V_2}{V_1 - V_o}, \quad \sigma'' \sim -V_2, \quad \sigma' - \sigma'' \sim \frac{V_o(V_1 - V_2)}{V_1 - V_o}$$

Because $V_1 \cdot \frac{V_o - V_2}{V_1 - V_o}$ is positive, $-V_2$ negative, one is tempted to believe that the upper branch will finally move in the "real" region $\sigma > 0$, the lower in the submerged region $\sigma < 0$. However, such assertions would be unwarranted since

V_2 may at times come too close to V_o and/or C. Yet as $\frac{V_o(V_1 - V_2)}{V_1 - V_o}$ always remains greater than V_o it is true that the two branches never come close to one another. Analysis uses the formulas

$$\overline{\lim}\ f(s) \leqq A, \quad \underline{\lim}\ f(s) \geqq B \quad (\text{for } s \to \infty)$$

to express the following behavior of a function f(s) : with any positive ϵ one can associate an ω such that $f(s) < A + \epsilon$ or $f(s) > B - \epsilon$ for all values $s > \omega$. Thus our relations imply

$$(28)\begin{cases} \overline{\lim}\ \sigma' \leqq \frac{V_o V_1}{V_1 - V_o}, \quad \underline{\lim}\ \sigma' \geqq 0 ; \\ \overline{\lim}\ \sigma'' \leqq 0, \ \underline{\lim}\ \sigma'' \geqq - V_o ; \\ \underline{\lim}\ (\sigma' - \sigma'') \geqq V_o . \end{cases}$$

In particular:

Supplement to Theorem 5. Owing to the hypothesis V, glancing incidence is impossible even in the limit for $s \to \infty$, in the sense that σ stays bounded while s approaches infinity.

In formula (26) the denominator $p_2^*(V_2 + \sigma)$ in the "real" region $\sigma \geqq 0$ is greater than $p_2^* V_2 \geqq p_2 V_2$. For ideal gases it is known that $p_2 V_2 = T_2$ tends to infinity with $s \to \infty$ on the Hugoniot line and that the temperature T_2 may be used as parameter on that line instead of s. Considering that σ stays bounded as $T_2 \to \infty$, we thus find $t \to 0$, $\sigma t' \to 0$, $\delta_1 \to 0$, and therefore the process of reflection is sure to approach head-on impact (and not its opposite,

glancing incidence) as $T_2 \to \infty$. Moreover it is easily seen that $N_2^* = p_2^* V_2^*$ is a monotone increasing function of T_2; and, under the assumption that the energy function $E = f(T)$ of the ideal gas tends to infinity with $T \to \infty$, N_2^* increases to infinity with $T_2 \to \infty$. Hence (25) reveals that there is only <u>one</u> branch of the (s,σ) contour $\mathcal{f}$ in the "real" region $\sigma \geqq 0$ as soon as T_2 exceeds a certain limit, namely, as soon as N_2^* has become greater than N_1^*. All this refers to ideal gases only. It does not seem safe to venture similar statements for arbitrary fluids; what we can predict here is the behavior of σ as exhibited by the supplement to Theorem 5 and the more precise formulas (28).

In physical reality the process of simple reflection as discussed in §§7 and 8 stops and gives way to a more complicated phenomenon as soon as s exceeds a certain limit. The problems relating to this <u>Mach phenomenon</u> are not yet solved, not even for an ideal gas with constant specific heat.

Part III. The Problem of the Shock Layer

9. Formulation of the problem

The fact that (1) is a total differential is expressed by the equation

$$\frac{\partial}{\partial p}\left(T\frac{\partial \eta}{\partial V} - p\right) = \frac{\partial}{\partial V}\left(T\frac{\partial \eta}{\partial p}\right)$$

or

$$\frac{\partial T}{\partial p}\cdot\frac{\partial \eta}{\partial V} - \frac{\partial T}{\partial V}\cdot\frac{\partial \eta}{\partial p} = 1. \tag{29}$$

For the sake of brevity we denote partial derivatives by subscripts as in $\frac{\partial \eta}{\partial p} = \eta_p$, $\frac{\partial \eta}{\partial V} = \eta_v$.

Our further argument will be based on the thermodynamical assumptions I-IV (§2), but in its course it will become clear that one more condition of highly plausible nature is to be added to our list:

Ia. Heating a quantum of fluid at constant volume raises its pressure and temperature.

It must be remembered that the analytic formulation (I'), (II') of the hypotheses I and II depended on the fixation of a sign. For the previous investigation this did not matter greatly, but it is essential for the behavior of the shock layer. We therefore require explicitly by the statement concerning pressure in Ia, that $\left(\frac{dp}{Td\eta}\right)_{V = \text{const.}} > 0$ or $\eta_p > 0$. The other part of Ia then adds the inequality $T_p/\eta_p > 0$ or $T_p > 0$. The requirement V that by heating at constant volume one can drive pressure up to arbitrarily high values will not be needed.

In a stationary field the state $(Z, \vec{u})$ of a fluid of given thermodynamical nature depends on the spatial coordinates x, y, z only, and not on time. The five equations which, in the absence of an external force, express conservation of mass, momentum and energy, take heat conductivity and viscosity into account by the flow of heat

$\vec{j} = -\lambda \cdot \text{grad } T$ and a stress tensor S of the form

$$S_{ik} = p\,\delta_{ik} - S^{*}_{ik}, \quad S^{*}_{ik} = \mu' \operatorname{div}\vec{u}\cdot\delta_{ik} + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

(in the writing down of which we have for a moment adopted the subscript notation). $\lambda = \lambda[Z]$, μ, μ' are given functions of the thermodynamical state Z; $\lambda > 0$, $\mu > 0$, $\mu' + \frac{2}{3}\mu > 0$, They are <u>small</u>. In two ways the passage to the limit $\lambda, \mu, \mu' \to 0$ can give rise to a discontinuity in the plane $x = 0$:

(A) Write ϵx for x and $\epsilon\lambda$, $\epsilon\mu$, $\epsilon\mu'$ for λ, μ, μ', leaving all other quantities untouched, and then let the positive constant factor ϵ tend to zero (<u>shock layer</u>).

(B) Let ϵ tend to zero after writing ϵx, ϵu for x, u and $\epsilon^2\lambda$, $\epsilon^2\mu$, $\epsilon^2\mu'$ for λ, μ, μ' (<u>slip layer</u>).

The different effect is clearly exemplified by the continuity equation

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{30}$$

Substitution (A) changes it into

$$\frac{\partial}{\partial x}(\rho u) + \epsilon\cdot\frac{\partial}{\partial y}(\rho v) + \epsilon\cdot\frac{\partial}{\partial z}(\rho w) = 0$$

which for $\epsilon \to 0$ reduces to

$$\frac{\partial}{\partial x}(\rho u) = 0.$$

whereas the process (B) leaves the three-term equation (30) unchanged. This is typical. None but the derivatives $\frac{\partial}{\partial x}$ survive the limiting process (A), the others $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ disappear from the equations. The shock layer is thus seen to be a one-dimensional problem, no interaction taking place between the several one-dimensional fibers of space y = const., z = const. The slip layer on the other hand constitutes a far more complicated three-dimensional problem, identical with that of Prandtl's boundary layer. We are here concerned with the shock layer only.

With the abbreviation $\mu^* = \mu' + 2\mu (> 0)$ the limiting process (A) leads to the following equations:

$$(31_a) \qquad \rho u = \text{const.} = M,$$

$$(31_b) \qquad Mu + p - \mu^* \frac{du}{dx} = \text{const.},$$

$$(31_c) \qquad Mv - \mu \frac{dv}{dx} = \text{const.}, \quad Mw - \mu \frac{dw}{dx} = \text{const.},$$

$$(31_d) \qquad \left(p - \mu^* \frac{du}{dx} + \rho E^*\right) u - \mu \left(v \frac{dv}{dx} + w \frac{dw}{dx}\right) - \lambda \frac{dT}{dx} = \text{const.},$$

where E^* denotes the total energy $= + \frac{1}{2}(u^2 + v^2 + w^2)$. Here the quantities λ, μ, μ' are no longer small. We assume that $(Z, \vec{u})$ approaches definite constant values $(Z_o, \vec{u}_o)$, $(Z_1, \vec{u}_1)$ for $x \to -\infty$ and $x \to +\infty$ respectively, whereas the derivatives $\frac{d}{dx}$ tend to zero.

The energy equation is equivalent to the entropy equation

$$(32) \qquad T \cdot \frac{d}{dx}\left(\rho u \eta - \frac{\lambda}{T} \frac{dT}{dx}\right) = \mu^* \left(\frac{du}{dx}\right)^2 + \mu \left\{\left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right\} + \frac{\lambda}{T} \left(\frac{dT}{dx}\right)^2,$$

from which one learns that the flow of entropy

$$\rho u \eta - \frac{\lambda}{T}\frac{dT}{dx} = M\eta - \frac{\lambda}{T}\frac{dT}{dx}$$

is a monotone increasing function of x and therefore $M\eta_1 \geqq M\eta_0$. The equality sign could prevail only if u, v, w, T and hence by (31_b) also p were constant. Excluding this trivial case of a constant state* $(Z, \vec{u})$ we thus obtain the strict inequality

$$M\eta_1 > M\eta_0 \tag{33}$$

implying $M \neq 0$. The latter remark shows how misleading it is to link the alternative $M \neq 0$, $M = 0$ to the discrimination of shock and slip layer: rather do these two possibilities correspond to the two ways (A) and (B) of passing to the limit of vanishing viscosity and heat conductivity. Being sure now that M does not vanish, we fall back upon our old relations connecting $(Z_0, \vec{u}_0)$ and $(Z_1, \vec{u}_1)$, including the Hugoniot relation. Let us assign the labels 0, 1 to the two states Z_0, Z_1, and accordingly orient the x-axis, so that $Z_0 < Z_1$. Then Theorem 2 combined with the law of increasing entropy (33) gives $M > 0$. The meaning of this inequality is that the shock front moves in the direction [1] → [0], or

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*Indeed, in general, p = const., T = const. imply V = const. There is, however, one extremely exceptional case in which the conclusion does not stand, namely if M = 0 and if for some constant value of p the temperature T as function of V stays constant in an entire V -interval. Maybe this is a warning against the danger of a singularity in the shock layer problem of which one steers clear with certainty only if one assumes $T_v \neq 0$ or, what is the same, if the hypotheses I-IV, Ia are preceded by one stating that p, T may serve as local parameters on $\mathfrak{Z}$ everywhere.

that Z_0 and not Z_1 is the initial state of the fluid not yet reached by the propagating shock.

10. Character of the two singular points Z_0, Z_1: Node and saddle*

It is easy to dispose of the equations (31_c) for v and w. Single out the one for v,

$$\mu\frac{dv}{dx} = M(v-b), \quad b = \text{const.},$$

and introduce the variable

$$\xi = M\int_0^x \frac{dx}{\mu(x)} \quad \{\mu(x) = \mu[Z(x)]\}.$$

We find

$$v(x) - b = C.e^{\xi}, \qquad C = \text{const.}$$

u(x) tends to positive values $u_0 = \mu[Z_0]$, $\mu_1 = \mu[Z_1]$ for $x \to -\infty$, $x \to +\infty$ respectively; therefore

$$\xi = \frac{M}{\mu_0}x + \sigma(x) \text{ for } x \to -\infty,$$
$$\xi = \frac{M}{\mu_1}x + \sigma(x) \text{ for } x \to +\infty.$$

Considering that $M > 0$ one realizes that v cannot tend to a finite limit v_1 for $x \to +\infty$ unless $C = 0$. Thus v and w must be constant throughout the shock layer.

(31_a) is satisfied by $u = MV$. Denoting by a/M the constant at the right of (31_b) and substituting from (31_b) the value of $p - \mu^* \frac{du}{dx}$ into (31_d) we are left, as is readily seen, with the following two ordinary differential equations of first order for the state $Z = Z(x)$:

$$(34)\begin{cases} \mu^* M.\dfrac{dV}{dx} = M^2(V - a) + b, \\ \dfrac{\lambda}{M}.\dfrac{dT}{dx} = E - \dfrac{1}{2}M^2(V - a)^2 - c. \end{cases}$$

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* For ideal gases of constant specific heat with certain special values of the constant χ, cf. R. Becker, Ztschr. f. Phys. 8 (1922), 321-362; in particular, pp. 339-347.

The constants M^2, a, c are determined by

$$M^2V_o + p_o = M^2V_1 + p_1 = M^2a,$$
$$E_o - \tfrac{1}{2}M^2(V_o - a)^2 = E_1 - \tfrac{1}{2}M^2(V_1 - a)^2 = c.$$

As the right members of (34) vanish for $Z = Z_o$ and Z_1, the two points Z_o, Z_1 turn out to be singular points for the differential system (34). If we care for the trajectory in the (p,V) -diagram only along which Z(x) moves while x runs from $-\infty$ to $+\infty$ we are faced with one ordinary differential equation of first order in the (p,V) -plane,

$$\frac{\lambda \cdot dT}{E - \tfrac{1}{2}M^2(V - a)^2 - c} = \frac{\mu^* \cdot dV}{(V - a) + (p/M^2)}.$$

But contrary to the customary viewpoint taken in the classical investigations on singular points, which were started by Briot and Bouquet almost one hundred years ago and are reproduced in all treatises on differential equations, our interest is in the parametrized trajectory Z = Z(x).

Let us briefly recount the well-known facts about singular points relevant for our purposes. Placing the singular point at the origin 0 we study two simultaneous differential equations for the "vector" $\varphi = (\varphi_1(x), \varphi_2(x))$ in a φ_1, φ_2 -plane,

$$\frac{d\varphi_1}{dx} = F_1(\varphi_1, \varphi_2), \quad \frac{d\varphi_2}{dx} = F_2(\varphi_1, \varphi_2)$$

where F_1, F_2 are given functions of φ_1, φ_2 vanishing at the origin. From the outset it is evident that any translation $x \longrightarrow x + h$ carries a solution $\varphi = (\varphi_1(x), \varphi_2(x))$ into a solution ("equivalent solutions"). Suppose the Taylor expansions of F_1, F_2 begin with the terms

$$F_1(\varphi_1, \varphi_2) \sim R_{11}\varphi_1 + R_{12}\varphi_2, \quad F_2(\varphi_1, \varphi_2) \sim$$

$$R_{21}\varphi_1 + R_{22}\varphi_2.$$

The equations

$$\varphi_1' = R_{11}\varphi_1 + R_{12}\varphi_2, \quad \varphi_2' = R_{21}\varphi_1 + R_{22}\varphi_2$$

define a linear mapping $\varphi \to \varphi' = R\varphi$. We determine its two eigen-values k and corresponding eigen-vectors $\varphi = \alpha$ (or rather eigen-d rections) by $R\alpha = k.\alpha$, explicitly

$$(k - R_{11})\alpha_1 - R_{12}\alpha_2 = 0,$$

$$- R_{21}\alpha_1 + (k - R_{22})\alpha_2 = 0,$$

and assume that the two roots k_1, k_2 of the secular equation

$$\begin{vmatrix} k - R_{11}, & - R_{12} \\ - R_{21}, & k - R_{22} \end{vmatrix} = 0$$

are real, distinct and $\neq 0$. In a suitable affine coordinate system in the φ_1, φ_2 -plane our mapping is then described by $\varphi_1' = k_1\varphi_1$, $\varphi_2' = k_2\varphi_2$, so that F_1, F_2 are of the form

$$F_1(\varphi_1, \varphi_2) = k_1\varphi_1 + \cdots,$$

$$F_2(\varphi_1, \varphi_2) = k_2\varphi_2 + \cdots .$$

We distinguish three cases according to the signs of the eigen values k_1, k_2. In the case (i) of two negative eigen-values, $k_2 < k_1 < 0$, every solution φ which comes sufficiently near to the origin plunges into it with x tending to $+\infty$; its asymptotic behavior is described by

$$\varphi_1(x) \sim a.e^{k_1 x}, \quad \varphi_2(x) \sim 0.e^{k_1 x} \quad (x \to \infty)$$

in the sense that

$$e^{-k_1 x} \cdot \varphi_1(x) = a + O(e^{-\epsilon x}),$$

$$e^{-k_1 x} \cdot \varphi_2(x) = o(e^{-\epsilon x})$$

where ϵ is a positive constant, the same for all solutions, while the constant a is specific (node, see Fig. 5). In the case (ii) of one negative, one positive eigen-value, $k_1 < 0$, $k_2 > 0$, there exist exactly two solutions (1_+, 1_- in Fig. 6) approaching 0 with $x \to +\infty$; their asymptotic behavior is described by

$$\varphi_1(x) \sim \pm e^{k_1 x}, \quad \varphi_2(x) \sim 0.e^{k_1 x} \quad (x \to +\infty)$$

(saddle). Of course this statement is to be interpreted so that equivalent solutions are considered as one and the same. The term saddle is better understood by observing that in this case (ii) two other solutions, 2_+ and 2_-, plunge into O in the direction of the φ_2-axis, while x goes to $-\infty$. In case (iii) of two positive eigen-values there is no solution whatsoever which approaches the origin as x tends to $+\infty$, except the trivial one $\varphi_1 = \varphi_2 = 0$ (whereas, of course, every solution that comes sufficiently near O takes the plunge when x moves toward $-\infty$).

With these facts in mind we now advance our decisive

Theorem 6. *For the differential equations of the shock layer determining the transition from a state* Z_0 *to a state* $Z_1 > Z_0$, *the initial state* Z_0 *is a node, the end state* Z_1 *a saddle.*

Clearly this behavior in the neighborhood of the two singular points Z_0, Z_1 is such as to favor the existence of a unique solution of the shock layer problem. All other

combinations would make one expect either no solution or an infinity of solutions.

Again, the proof depends, above all, on the fundamental inequalities (14) for $M^2 = (p_1 - p_0)/(V_0 - V_1)$ which, η_v/η_p being the adiabatic derivative $m = -\frac{dp}{dV}$ we may write in the form

$$(35) \qquad M^2 \eta_p^{(0)} - \eta_v^{(0)} > 0, \quad M^2 \eta_p^{(1)} - \eta_v^{(1)} < 0.$$

They must be combined with

$$(36) \qquad M > 0; \quad \eta_p > 0, \quad T_p > 0$$

First investigate the neighborhood of Z_0 and therefore write

$$M^2(V - a) + p = M^2(V - V_0) + (p - p_0).$$

For infinitesimal $\delta p = p - p_0$, $\delta V = V - V_0$

$\delta E = T_0 \delta\eta - p_0 \delta V$, $\delta \frac{1}{2}(V - a)^2 = (V_0 - a)\delta V$, hence

$$\delta\left\{E - \tfrac{1}{2}M^2(V - a)^2\right\} = T_0\delta\eta - \left\{p_0 + M^2(V_0 - a)\right\}\delta V = T_0\delta\eta ,$$

therefore in neglecting terms of higher order

$$E - \tfrac{1}{2}M^2(V - a)^2 - c \sim T_0(\eta - \eta_0).$$

The following approximate linear system for δV, δp results:

$$\frac{d}{dx}(\delta V) = \gamma_0(M^2 \delta V + \delta p),$$

$$\frac{d}{dx}(\delta T) = \gamma_0' \cdot \delta\eta$$

where

$$\gamma = 1/M\mu^* > 0, \quad \gamma' = TM/\lambda > 0$$

and $\delta\eta$, δT stand for

$$\delta\eta = \eta_p^{(0)} \delta p + \eta_v^{(0)} \delta V, \quad \delta T = T_p^{(0)} \delta p + T_v^{(0)} \delta V.$$

In order to avoid unnecessary encumbrance, we drop the index 0. Then the equations determining the eigen-values k and corres-

ponding eigen-directions (δV, δp) at Z_o become

$$k.\,\delta V = \gamma(M^2\,\delta V + \delta p), \quad k.\,\delta T = \gamma'.\,\delta\eta$$

giving rise to the following secular equation for k,

$$\begin{vmatrix} k - \gamma M^2, & -\gamma \\ kT_v - \gamma'\eta_v, & kT_p - \gamma'\eta_p \end{vmatrix} = 0 \qquad \text{or}$$

(37) $$T_p.k^2 - [\gamma(M^2T_p - T_v) + \gamma'\eta_p].k + \gamma\gamma'(M^2\eta_p - \eta_v) = 0$$

Compute the discriminant

$$\Delta = \{\gamma(M^2T_p - T_v) + \gamma'\eta_p\}^2 - 4\gamma\gamma'T_p(M^2\eta_p - \eta_v) =$$
$$\{\gamma(M^2T_p - T_v) - \gamma'\eta_p\}^2 + 4\gamma\gamma'(T_p\eta_v - T_v\eta_p)$$

and thus verify by (29) that Δ is positive,

$$\Delta = \{\gamma(M^2T_p - T_v) - \gamma'\eta_p\}^2 + 4\gamma\gamma' \geqq 4\gamma\gamma' = \frac{4T}{\lambda\mu^*} > 0.$$

Consequently the eigen-values are real and distinct, at the singular point Z_o as well as at Z_1.

Their product equals

$$(M^2\eta_p - \eta_v)/T_p,$$

and because of (36) and the fundamental inequalities (35) that product is positive at Z_o, negative at Z_1. Hence the two eigen values are of opposite sign at Z_1, of equal sign at Z_o. Whether the latter sign is positive or negative can be decided by the sum of the two k's for which (37) gives the value

$$\{\gamma(M^2T_p - T_v) + \gamma'\eta_p\}/T_p.$$

All constituents of this expression are positive, including

$$M^2T_p^{(o)} - T_v^{(o)} > \frac{\eta_v^{(o)}}{\eta_p^{(o)}} \cdot T_p^{(o)} - T_v^{(o)} = 1/\eta_p^{(o)} > 0.$$

Thus we arrive at the result that the two roots k at Z_o are positive. Our theorem has now been proved, and a third aspect

of the fundamental inequalities (13) has been disclosed: Not only do they rule that by a shock the flow changes from supersonic to subsonic and that glancing incidence of a shock wave reflected by a wall is forbidden, but they also decide that one of the two points Z_o, Z_1 is a node, the other a saddle.

In order actually to prove the existence of a unique solution of the shock layer problem, one has to bridge the gap between the neighborhoods of the two singular points Z_o, Z_1 by some sort of topological argument. In that respect one fact is of decisive importance:

Theorem 7. The differential equations of the shock layer have no singular point besides Z_o, Z_1.

Indeed if there existed three distinct singular points Z_o, Z_1, Z_2 they would satisfy the equations

$$M^2V_o + p_o = M^2V_1 + p_1 = M^2V_2^2 + p_2 = M^2a,$$

$$E_o - \tfrac{1}{2}M^2(V_o - a)^2 = E_1 - \tfrac{1}{2}M^2(V_1 - a)^2 = E_2 - \tfrac{1}{2}M^2(V_2 - a)^2.$$

The first set states that Z_o, Z_1, Z_2 lie on a straight line of negative inclination $-M^2$, and we may assume the arrangement $Z_o < Z_1 < Z_2$. Then the second set requires that both $Z = Z_1$ and Z_2 satisfy the Hugoniot equation $H(Z\ Z_o) = 0$, in contradiction to a statement previously proved by which there cannot be more than one point Z on any ray from Z_o satisfying that equation. The observation is urged upon us

that whereas the Hugoniot relation is not transitive, the relation between Z_o, Z_1 defined by the simultaneous equations

$$M^2V_o + p_o = M^2V_1 + p_1, \qquad H(Z_1Z_o) = 0$$

is.(M^2 being a given constant).

The topological picture of the trajectories which one expects is indicated by Fig. 7. If this is correct, the region covered by all trajectories running into Z_o for $x \longrightarrow -\infty$ would be bounded by the saddle line through Z_1 consisting of the two branches 2_+ and 2_- (Fig. 6). The outlook is encouraging.

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